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LOCAL BUCKLING OF DELAMINATED SANDWICH BEAMS USING CONTINUOUS ANALYSIS

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Abstract—The paper presents a method of continuous analysis for predicting the local delamination buckling load of the face sheet of sandwich beams. The discontinuous system of a face sheet delaminated from the elastic core is treated as a continuous system of the face sheet without debond but subjected to an added force system that causes the net interfacial tractions at the separated region to vanish. The effect of transverse normal and shear resistance from the core is accounted for. The procedure allowing direct determination of the buckling load by considering the entire region without separating it into regions with and without delaminations is effective for this class of problems. Fourier series in conjunction with the Stokes transformation is used which provides a unified solution for problems with different end conditions. The concept of analysis can foreseeably be extended to two-dimensional plate problems with delamination regions of arbitrary shapes. Some numerical results are presented to illustrate the applicability of the analysis procedure and effects of various parameters to the buckling load. Copyright © 1996 Elsevier Science Ltd.

INTRODUCTION

Debonding of face sheets from the core is one of the most common failure modes in sandwich structures. Delaminations may occur due to various reasons such as manufacturing imperfections, or impact of foreign objects. Under compression load, the delaminated structure may buckle. Delaminated composite sandwich beams have gained renewed interest though sandwich construction has been used in various structural applications for many years due to its light weight and high bending rigidity, etc. For composite sandwiches, the interface between the face and core may be weaker than those in layered composite laminates. Generally speaking, the delaminated composite sandwich beams referred only to the sandwich beams for which debonds exist in the interface between the face and core. Under bending, one of the face sheets of the sandwich beam is under compression and the other one is under tension. If there is a debond, local buckling of the face sheet may occur. Since Chai *et al.* (1981) established an analytic one-dimensional model for the analysis of beam-plates delamination buckling, the delamination buckling of a one-dimensional beam-plate has been studied by several researchers, such as Simitses *et al.* (1985), Yin *et al.* (1986), Kardomateas and Schmueser (1988). All of these studies considered that the two parts at the delaminated region are completely detached from each other. Although there are studies dealing with the buckling of composite sandwich beams, such as the works of Rao *et al.* (1985, 1986), Mingust *et al.* (1988), Frostig *et al.* (1992, 1993), most of the earlier investigations do not involve delamination. The analysis for the buckling of delaminated sandwich beams has been made by some authors in recent years. Somer *et al.* (1991) developed a theoretical model based on the earlier work for Chai *et al.* (1981) to study the local buckling of delaminated sandwich beams, and Frostig (1992) described the behavior of a general sandwich beam with a delamination at one of the skin-core interfaces for stress analysis. Hwu *et al.* (1992) developed a one-dimensional model to analyze the overall buckling of the delaminated sandwich beams. The present study presents a continuous analysis procedure for determining the local buckling load for the face sheet debonded from the core. The general concept of the continuous analysis for discontinuous structures

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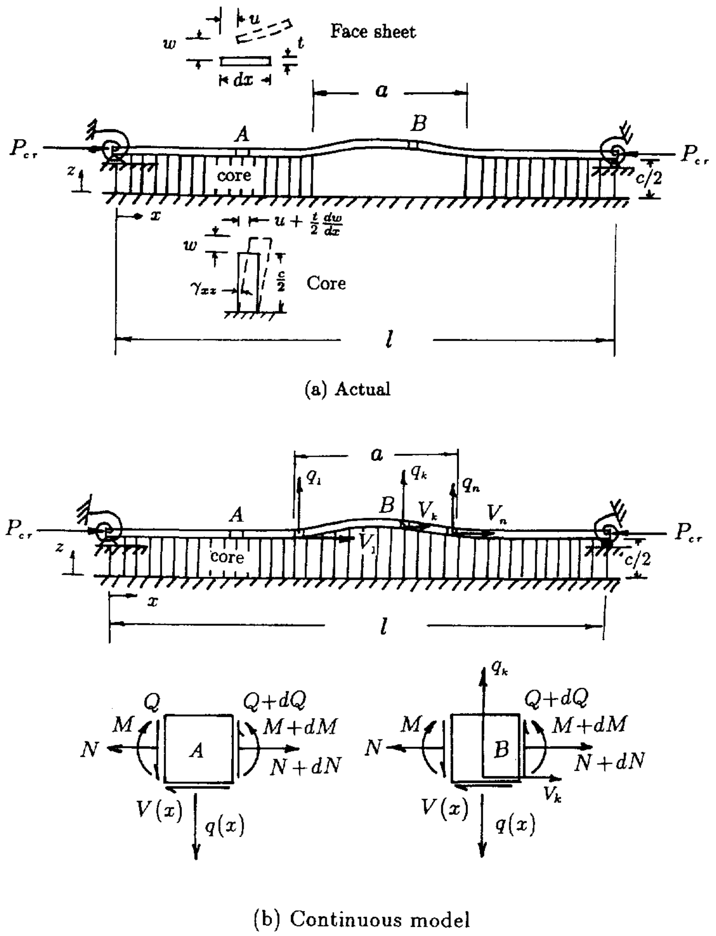

Fig. 1. Buckled face sheet.

was presented by Wang (1993). The core is considered to provide transverse normal as well as shear resistance to the face sheet in the bonded region. While the transverse displacement at both ends of the face sheet is considered to be zero in this study for the convenience of discussion, rotational restraints are included. The analysis is, therefore, quite general and may be extended to two-dimensional problems for delaminated plates having various boundary conditions.

MODEL

A sandwich beam of unit width with orthotropic composite face sheets is considered in the study. The model considers that a through-the-width debond exists at the interface between the face and the core. Only the transverse stiffness normal to the face sheets and transverse shear stiffness are considered in the core. As a result, the face sheets take all of the in-plane loading, whereas the core takes only transverse shear and normal forces as indicated in Fig. 1. We consider a delaminated sandwich beam which has an interface debond occurring between the upper face and the core. When the beam is under pure bending, the upper face takes a compressive axial load and the lower face takes a tensile axial load. Since we are only interested in the local delamination buckling in the study, the buckled configuration of the upper half of the sandwich beam, as shown in Fig. 1, is considered. In the model illustrated in Fig. 1, the foundation is modeled by parallel linear and shear springs distributed along the bonded interface between the face sheet and the core. They are characterized by the distributions of shear force $V(x)$ and normal force $q(x)$ in force per unit area of the unit width beam. Figure 1a shows the actual buckled face sheet, and Fig. 1b shows the continuous model which is mechanically equipollent to the actual structure. Following the idea of the continuous analysis discussed by Wang (1993), we consider that any point at the interface of the debonded region for the continuous model as shown in Fig. 1b is also attached to the core but with added fictitious normal and shear forces q_j and V_j per unit width concentrated at the jth point ($x = x_j$) in the delaminated region so that the net traction at the point vanishes during buckling. Accordingly, we divide the debonded region into n divisions and require that the conditions $q_j = k_f w(x_j)\Delta x_j$ and $V_j = V(x_j)\Delta x_j$ be satisfied over each division of Δx_j length in the debonded region where k_f is the transverse normal stiffness of the core. $V(x)$ is the interface shear force which is equal to $G_{xz}\gamma_{xz}$ of the core where G_{xz} and γ_{xz} are, respectively, the shear modulus and shear strain of the core which are considered to be uniform through the thickness of the core. The face sheet is considered to be supported with zero displacement for the convenience of discussion but with rotational restraint in general at each end as shown in Fig. 1.

Governing equations

The equilibrium equations governing the bifurcation buckling of a unit width face sheet attached to the core having transverse normal as well as shear resistance are

$$\frac{dN}{dx} - V + \sum_{k=1}^{n} V_k\delta(x-x_k) = 0 \tag{1}$$

$$\frac{d^2M}{dx^2} - P\frac{d^2w}{dx^2} + \left(\frac{t}{2}\right)\frac{dV}{dx} = -\sum_{k=1}^{n} q_k\delta(x-x_k) + q(x) + \left(\frac{t}{2}\right)\left[\sum_{k=1}^{n} V_k\delta'(x-x_k)\right] \tag{2}$$

where P is the buckling load and t is the thickness; N is the axial force and M is the bending moment in the face sheet, q and V are transverse normal and shear resistance from the core, respectively, during buckling of the unit width beam-plate; x is the coordinate in the longitudinal direction, and $\delta(x-x_k)$ is the Dirac delta function.

For symmetrically stacked orthotropic laminates for the face sheets, the constitutive equations and strain-displacement relations for the face sheet are

$$N = A_{11}\varepsilon_x, \quad M = D_{11}\kappa_x \tag{3}$$

and

$$\varepsilon_x = \frac{du}{dx}, \quad \kappa_x = -\frac{d^2w}{dx^2} \tag{4}$$

where A_{11} and D_{11} are extensional and flexural stiffnesses, respectively, ε_x is the extensional strain, κ_x is the change of curvature and u is the longitudinal displacement. By substituting eqns (3) and (4) into eqns (1) and (2), we obtain

$$A_{11}\frac{d^2u}{dx^2} - V = -\sum_{k=1}^{n} V_k\delta(x-x_k) \tag{5}$$

$$D_{11}\frac{d^4w}{dx^4} + P\frac{d^2w}{dx^2} - \left(\frac{t}{2}\right)\frac{dV}{dx} + q(x) = \sum_{k=1}^{n} q_k\delta(x-x_k) - \left(\frac{t}{2}\right)\left[\sum_{k=1}^{n} V_k\delta'(x-x_k)\right] \tag{6}$$

The transverse resistance from the core of the sandwich beam is taken in the usual form

$$q(x) = k_f w \tag{7}$$

in which

$$k_f = \frac{1}{c} E_c A_{eff} \tag{8}$$

where E_c is the Young's modulus of the core in the transverse z-direction, A_{eff} is the effective contact area of, e.g., the honeycomb, core per unit area of the face sheet, and c is the depth of the core. The tangential resistance from the core of the sandwich beam is taken to be

$$V(x) = G_{xz}\gamma_{xz} \tag{9}$$

where G_{xz} is the transverse shear modulus of the core, and the shear strain in the core is

$$\gamma_{xz} = \left(u + \frac{t}{2}\frac{\mathrm{d}w}{\mathrm{d}x}\right) \Big/ \left(\frac{c}{2}\right) \tag{10}$$

By substituting eqns (7)–(10) into eqns (5) and (6), we arrive at the following governing differential equations:

$$A_{11}\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} - \frac{2G_{xz}}{c}\left[u + \left(\frac{t}{2}\right)\frac{\mathrm{d}w}{\mathrm{d}x}\right] = -\sum_{k=1}^{n} V_k\delta(x-x_k) \tag{11}$$

$$D_{11}\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} + P\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} - \left(\frac{tG_{xz}}{c}\right)\left[\frac{\mathrm{d}u}{\mathrm{d}x} + \left(\frac{t}{2}\right)\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right] + k_f w = \sum_{k=1}^{n} q_k\delta(x-x_k) - \left(\frac{t}{2}\right)\sum_{k=1}^{n} V_k\delta'(x-x_k) \tag{12}$$

If the shear stiffness of the core is negligible, eqns (11) and (12) for the buckling of the face sheet per unit width are reduced to

$$D_{11}\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} + P\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + k_f w = \sum_{k=1}^{n} q_k\delta(x-x_k) \tag{13}$$

which is the well known equation for beam-columns on the Winkler foundation.

Boundary conditions

If both ends of the face sheet are simply supported, the boundary conditions are

$$\text{at } x = 0: \quad w(0) = w''(0) = u'(0) = 0 \tag{14}$$

$$\text{at } x = l: \quad w(l) = w''(l) = u'(l) = 0 \tag{15}$$

where primes denote differentiations with respect to x, and l is the total length of the face sheet. For the general case of hinge support with rotational restraint at both ends, the boundary conditions are

$$\text{at } x = 0: \quad w(0) = u'(0) = 0, \quad \text{and } D_{11}w''(0) = K_0 w'(0) \tag{16}$$

$$\text{at } x = l: \quad w(l) = u'(l) = 0, \quad \text{and } D_{11}w''(0) = -K_1 w'(l) \tag{17}$$

where K_0 and K_1 are rotational spring constants at $x = 0$ and l, respectively. If the shear stiffness of the core is negligible, the boundary conditions without the involvement of u in eqns (14) and (17) are used with the governing differential eqn (13).

Analytical solutions

For general boundary conditions at both ends of the face sheet, Fourier series in conjunction with the Stokes transformation discussed by Chuang and Wang (1991), and Chung (1981), along with various series properties given by Bromwich (1965) is used. The displacements are represented by the following Fourier series

$$u = u_0 + \sum_{m=1}^{\infty} u_m \cos \alpha_m x \tag{18}$$

$$w(x) = \begin{cases} \sum_{m=1}^{\infty} w_m \sin \alpha_m x & 0 < x < l \\ B_0 & x = 0 \\ B_1 & x = l \end{cases} \tag{19}$$

where $\alpha_m = m\pi/l$. Their derivatives are

$$\frac{du}{dx} = -\sum_{m=1}^{\infty} \alpha_m u_m \sin \alpha_m x \tag{20}$$

$$\frac{dw(x)}{dx} = \frac{(B_1 - B_0)}{l} + \sum_{m=1}^{\infty} \bar{w}_m \cos \alpha_m x \quad 0 \leqslant x \leqslant l \tag{21}$$

with

$$\bar{w}_m = \frac{2}{l}[B_1(-1)^m - B_0] + \alpha_m w_m$$

$$\frac{d^2 w(x)}{dx^2} = \begin{cases} -\sum_{m=1}^{\infty} \alpha_m \bar{w}_m \sin \alpha_m x & 0 < x < l \\ B_0'' & x = 0 \\ B_1'' & x = l \end{cases} \tag{22}$$

$$\frac{d^3 w}{dx^3} = \frac{(B_1'' - B_0'')}{l} + \sum_{m=1}^{\infty} \left\{ \frac{2}{l}[B_1''(-1)^m - B_0''] - \alpha_m^2 \bar{w}_m \right\} \cos \alpha_m x \quad 0 \leqslant x \leqslant l \tag{23}$$

$$\frac{d^4 w}{dx^4} = -\sum_{m=1}^{\infty} \alpha_m \left\{ \frac{2}{l}[B_1''(-1)^m - B_0''] - \alpha_m^2 \bar{w}_m \right\} \sin \alpha_m x \quad 0 < x < l \tag{24}$$

The Dirac delta function and its derivative are also represented by Fourier cosine series and/or sine series,

$$\delta(x - x_k) = \sum_{m=1}^{\infty} D_m \sin \alpha_m x \tag{25}$$

$$\delta(x - x_k) = \frac{C_0}{2} + \sum_{m=1}^{\infty} C_m \cos \alpha_m x \tag{26}$$

$$\delta'(x-x_k) = -\sum_{m=1}^{\infty} \alpha_m C_m \cos \alpha_m x_k \sin \alpha_m x \tag{27}$$

where

$$D_m = \tfrac{2}{l} \sin \alpha_m x_k$$

$$C_0 = \tfrac{2}{l} \quad \text{and} \quad C_m = \tfrac{2}{l} \cos \alpha_m x_k$$

Since we consider that the face sheet has zero transverse displacement at both ends for the convenience of discussion, hence, $B_0 = B_l = 0$. By substituting eqns (18)–(27) into eqns (11) and (12), we obtain

$$u_0 = \sum_{k=1}^{n} \frac{1}{2} \frac{\bar{V}_k}{\bar{G}} \tag{28}$$

$$a_{11} u_m + a_{12} w_m = 2 \sum_{k=1}^{n} \bar{V}_k \cos \alpha_m x_k \tag{29}$$

$$a_{21} u_m + a_{22} w_m = \sum_{k=1}^{n} \bar{V}_k \bar{t} m\pi \cos \alpha_m x_k + 2 \sum_{k=1}^{n} \bar{q}_k \sin \alpha_m x_k \tag{30}$$

where

$$a_{11} = 2\bar{G} - m^2 \bar{A} \qquad a_{12} = \bar{G}\bar{t}m\pi$$

$$a_{21} = \bar{G}\bar{t}m\pi \qquad a_{22} = m^4 + \bar{K} - (\bar{P} - \tfrac{1}{2}\bar{G}\bar{t}^2\pi^2)m^2$$

$$\bar{A} = \frac{A_{11} l^2}{D_{11}\pi^2} \qquad \bar{G} = \frac{G_{xz} l^4}{D_{11} c \pi^4}$$

$$\bar{V}_k = \frac{V_k l^3}{D_{11}\pi^4} \qquad \bar{K} = \frac{k_f l^4}{D_{11}\pi^4}$$

$$\bar{P} = \frac{P}{P_{Euler}} \qquad \bar{q}_k = \frac{q_k l^3}{D_{11}\pi^4}$$

$$\bar{t} = \frac{t}{l} \qquad P_{Euler} = \frac{D_{11}\pi^2}{l^2}$$

If both ends of the face sheet are simply supported, then $B_0'' = B_l'' = 0$. If the shear stiffness of the core is negligible, eqns (28), (29) and (30) are reduced to

$$w_m = \frac{2}{\lambda_m l}\left(\frac{l}{\pi}\right)^4 [(-1)^m \alpha_m B_l'' - \alpha_m B_0''] + \frac{2}{\lambda_m} \sum_{i=1}^{n} \bar{q}_i \sin \alpha_m x_i \tag{31}$$

in which

$$\lambda_m = m^4 + \bar{K}\left(\frac{l}{\pi}\right)^4 - \frac{P}{P_{Euler}} m^2$$

and the Euler buckling load per unit width of the beam-plate is

$$P_{Euler} = \frac{D_{11}\pi^2}{l^2}$$

By solving eqns (29) and (30) for the general case, we obtain

$$u_m = \sum_{k=1}^{n} A_{mk} V_k + \sum_{k=1}^{n} B_{mk} q_k + (-1)^m \alpha_m a_m B_1'' + \alpha_m b_m B_0'' \tag{32}$$

$$w_m = \sum_{k=1}^{n} C_{mk} V_k + \sum_{k=1}^{n} D_{mk} q_k + (-1)^m \alpha_m c_m B_1'' + \alpha_m d_m B_0'' \tag{33}$$

where

$$A_{mk} = \frac{D^*}{D}(2a_{22} \cos \alpha_m x_k - a_{12} \bar{t} m\pi \cos \alpha_m x_k)$$

$$B_{mk} = \frac{D^*}{D}(-2a_{12} \sin \alpha_m x_k)$$

$$C_{mk} = \frac{D^*}{d}(a_{11} \bar{t} m\pi \cos \alpha_m x_k - 2a_{21} \cos \alpha_m x_k)$$

$$D_{mk} = \frac{D^*}{D}(2a_{11} \sin \alpha_m x_k)$$

$$D^* = \frac{l^3}{D_{11}\pi^4}$$

$$a_m = -\frac{2l^3}{D\pi^4} a_{12}$$

$$b_m = -a_m$$

$$c_m = \frac{2l^3}{D\pi^4} a_{11}$$

$$d_m = -c_m$$

$$D = a_{11}a_{22} - a_{12}a_{21}$$

Now, we have the general solutions for u and w from which we obtain $q(x)$ from eqn (7) and $V(x)$ from eqn (9). In the debonded region, we require the following conditions to be satisfied at $x = x_j$:

$$V(x_j)\Delta x = V_j \tag{34}$$

$$q(x_j)\Delta x = q_j \tag{35}$$

where $j = 1, 2, 3, \ldots n$, and n is the number of divisions associated with n sets of fictitiously added force components in the debonded region. The length of each division, Δx, is considered to be constant in this study. If both ends of the face sheet are simply supported, eqns (32) and (33) with $B_1'' = B_0'' = 0$ satisfying all the boundary conditions become the general solutions of the problem. As a result, conditions (34) and (35) are sufficient to establish the eigenvalue problem for determining the delamination buckling load. If either one or both ends are rotationally restrained, the third condition given in eqns (16) and/or (17) must be considered additionally. For the general case that both ends are rotationally restrained, the two conditions $D_{11}B_1'' = -K_1 w'(l)$ and $D_{11}B_0'' = K_0 w'(0)$ together with conditions (34) and (35) result in the following equations after eqns (7), (9), (19), (21), (32), and (33) are used where applicable:

$$[A^*_{jk}]\{V_k\}+[B^*_{jk}]\{q_k\}+\{R_{j,n-1}\}B_1''+\{R_{j,n+2}\}B_0''=0 \tag{36}$$

$$[C^*_{jk}]\{V_k\}+[D^*_{jk}]\{q_k\}+\{S_{j,n+1}\}B_1''+\{S_{j,n+2}\}B_0''=0 \tag{37}$$

$$[A^*_{n+1,k}]\{V_k\}+[B^*_{n+1,k}]\{q_k\}+\{R_{n+1,n+1}\}B_1''+\{R_{n+1,n+2}\}B_0''=0 \tag{38}$$

$$[C^*_{n+1,k}]\{V_k\}+[D^*_{n+1,k}]\{q_k\}+\{S_{n+1,n+1}\}B_1''+\{S_{n+1,n+2}\}B_0''=0 \tag{39}$$

where

$$A^*_{jk}=\left[\frac{1}{2}(\Delta\bar{x}-\delta_{jk})+\bar{G}\Delta\bar{x}\sum_{m=1}^{\infty}\left(A_{mk}+\frac{1}{2}m\pi\bar{t}C_{mk}\right)\cos\alpha_m x_j\right]D^*$$

$$B^*_{jk}=\left[\bar{G}\Delta\bar{x}\sum_{m=1}^{\infty}\left(B_{mk}+\frac{1}{2}m\pi\bar{t}D_{mk}\right)\cos\alpha_m x_j\right]D^*$$

$$C^*_{jk}=\left[\bar{K}\Delta\bar{x}\sum_{m=1}^{\infty}C_{mk}\sin\alpha_m x_j\right]D^*$$

$$D^*_{jk}=\left[-\delta_{jk}+\bar{K}\Delta\bar{x}\sum_{m=1}^{\infty}D_{mk}\sin\alpha_m x_j\right]D^*$$

$$R_{j,n+1}=\bar{G}\Delta\bar{x}\sum_{m=1}^{\infty}m\pi(-1)^m\left(a_m+\frac{1}{2}m\pi\bar{t}c_m\right)\cos\alpha_m x_j$$

$$R_{j,n+2}=\bar{G}\Delta\bar{x}\sum_{m=1}^{\infty}m\pi\left(b_m+\frac{1}{2}m\pi\bar{t}d_m\right)\cos\alpha_m x_j$$

$$S_{j,n+1}=\bar{K}\Delta\bar{x}\sum_{m=1}^{\infty}m\pi(-1)^m c_m\sin\alpha_m x_j$$

$$S_{j,n+2}=\bar{K}\Delta\bar{x}\sum_{m=1}^{\infty}m\pi d_m\sin\alpha_m x_j$$

$$\Delta\bar{x}=\frac{\Delta x}{l}$$

$$A^*_{n,+1,k}=D^*\sum_{m=1}^{\infty}(-1)^m\alpha_m C_{mk}$$

$$B^*_{n+1,K}=D^*\sum_{m=1}^{\infty}(-1)^m\alpha_m D_{mk}$$

$$C^*_{n+1,k}=D^*\sum_{m=1}^{\infty}\alpha_m C_{mk}$$

$$D^*_{n+1,k}=D^*\sum_{m=1}^{\infty}\alpha_m D_{mk}$$

$$R_{n+1,n+1}=\frac{D_{11}}{K_1}+\sum_{m=1}^{\infty}\alpha_m^2 c_m$$

$$R_{n+1,n+2} = \sum_{m=1}^{\infty} (-1)^m \alpha_m^2 d_m$$

$$S_{n+1,n+1} = \sum_{m=1}^{\infty} (-1)^m \alpha_m^2 c_m$$

$$S_{n+1,n+2} = -\frac{D_{11}}{K_0} + \sum_{m=1}^{\infty} \alpha_m^2 d_m$$

Equations (36)–(39) may be written in the following matrix form:

$$[A]\{Y\} = 0 \tag{40}$$

where

$$\{Y\} = [\{V_k\}\{q_k\}B_1'' \, B_0'']^{\mathrm{T}}.$$

The delamination buckling load may be determined by requiring the determinant of the coefficient matrix of eqn (40) to vanish, while eqns (36)–(39) allow one to solve for the general case of delamination buckling load of the face sheet accounting for transverse normal and shear resistance of the core and the rotational restraints at both ends. However, if certain effects do not exist, it is preferable to use reduced equations instead of assigning very small or large values for related restraining parameters. The following identifies the special cases:

1. If the face sheet is simply supported at $x = 0$, we should delete terms involving B_0'' and use eqns (36)–(38).
2. If the face sheet is simply supported at $x = l$, we should delete terms involving B_1'' and use eqns (36), (37), and (39).
3. If both ends of the face sheet are simply supported, we should delete terms involving B_0'' and B_1'' and use eqns (36) and (37).
4. If the face sheet is fixed at $x = 0$ and/or $x = l$, D_{11}/K_0 and/or D_{11}/K_1 in $S_{n+1,n+2}$ and/or $R_{n+1,n+1}$ should be taken as zeros respectively.
5. If the transverse shear stiffness in the core is negligible, eqns (36) and (37) are reduced to a single set of equations as follows:

$$[\zeta_{ij}]\{q_j\} - S_i^* B_0'' + R_i^* B_1'' = 0 \tag{41}$$

Equations (38) and (39) are reduced to

$$-\sum_{i=1}^{n} S_i^* q_i + b_{11} B_0'' - b_{12} B_1'' = 0 \tag{42}$$

$$\sum_{i=1}^{n} R_i^* q_i - b_{21} B_0'' + b_{22} B_1'' = 0 \tag{43}$$

where

$$\zeta_{ij} = -\frac{l}{2\bar{K}\Delta x}\delta_{ij} + \sum_{m=1}^{\infty} \frac{1}{\lambda_m} \sin \alpha_m x_i \sin \alpha_m x_j$$

$$S_i^* = D^* \sum_{m=1}^{\infty} \frac{m}{\lambda_m} \sin \alpha_m x_i$$

$$R_i^* = D^* \sum_{m=1}^{\infty} (-1)^m \frac{m}{\lambda_m} \sin \alpha_m x_i$$

$$S = \sum_{m=1}^{\infty} \frac{m^2}{\lambda_m}$$

$$R = \sum_{m=1}^{\infty} (-1)^m \frac{m^2}{\lambda_m}$$

$$b_{11} = S + \frac{1}{2\bar{K}_0}$$

$$b_{12} = b_{21} = R$$

$$b_{22} = S + \frac{1}{2\bar{K}_1}$$

$$\bar{K}_0 = \frac{K_0 l}{D_{11}\pi^2}$$

$$\bar{K}_1 = \frac{K_1 l}{D_{11}\pi^2}$$

Equations (41)–(43) may be written in the following matrix form:

$$[\bar{A}]\{\bar{Y}\} = 0 \tag{44}$$

where $\{\bar{Y}\} = [q_j \; B_0'' \; B_1'']^{\mathrm{T}}$, and the elements of the matrix $[\bar{A}]$ can be identified from eqns (41)–(43). The delamination buckling load can be determined by setting the determinant of the coefficient matrix of eqn (44) to zero. If there is no debond between the face sheet and the core, the buckling determinant reduces to

$$b_{11}b_{22} - b_{12}b_{21} = 0 \tag{45}$$

If the face sheet is entirely delaminated from the core, the buckling load can be determined from eqn (45) by taking $\bar{K} = 0$. If both ends of the face sheet are also simply supported, the buckling load reduces to the Euler's load by setting $\lambda_m = 0$ with $m = 1$ and $\bar{K} = 0$.

NUMERICAL RESULTS

Buckling loads without effect of shear stiffness of the core

A few degenerated cases of beam-plates having zero and full delaminations where exact solutions are available are considered for demonstrating the accuracy of the present method of continuous analysis using very small and large delamination lengths, respectively. They also serve the purpose of partially checking the derivation and computer programs. For all these cases, a beam of unit width having $l = 31.4$ inches and $D_{11} = 100$ lb-in are used. The effect of the rotational restraints at end supports is also examined using the simplified model where the shear stiffness of the core is neglected for a beam-plate having various delamination lengths. Computer programs for a face sheet with both ends simply supported using eqn (41) with B_0'' and B_1'' omitted, and hinge supports with rotational restraints using eqn (44) are written separately for comparison purposes.

For the case where the shear resistance of the core is neglected, the problem becomes a beam on the elastic Winkler foundation. The buckling load of 200-lb is given in the example on page 36 of the book by Brush and Almroth (1975) for a simply supported beam on a continuous foundation with $k_f = 100$ lb/in^3. The converged result obtained from the present continuous analysis by considering a very small central debond length of $a = 0.003l$

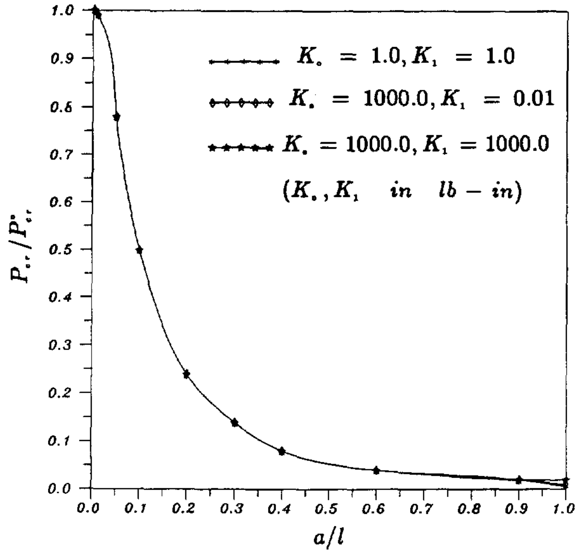

Fig. 2. Effect of end rotational restraint for $k_f = 100$ lb/in^3 for a unit width beam having a central debond.

using the computer program for both ends simply supported face sheet of unit width is found to be 199.99-lb. For the same beam on a very soft foundation of $k_f = 0.01$ lb/in^3, the buckling load for a face sheet having $a = 0.97l$ central debond length is found to be 1.001-lb which coincides with the Euler's buckling load for the pin ended beam-plate of unit width.

To give some indications of the effectiveness of the Stokes transformation, the buckling load obtained by using the program for hinged beams with rotational restraints, the buckling load for $k_f = 0.01$ lb/in^3 with very small rotational spring constants of $K_0 = K_1 = 1.0$ lb-in is found to be 1.001 lb per unit plate width. The result coincides with Euler's load, and is also consistent with the result given earlier using the computer program for simply supported face sheet. For $K_0 = 1000$ lb-in and $K_1 = 0.01$ lb-in, the present result for the buckling load is found to be 2.044 lb per unit plate width which coincides with the well-known buckling load of a clamped-simply supported column without foundation. For $K_0 = K_1 = 1000$ lb-in, the present result is found to be 4.003 lb per unit plate width which coincides with the well known buckling load of a clamped-clamped column without foundation. For illustrating the applicability of the continuous analysis procedure for discontinuous structures having delaminations in conjunction with Stokes transformation to account for end rotational restraints in the Fourier series representation of deflection, buckling loads of the face sheet for $k_f = 100$ lb/in^3 with various combinations of K_0 and K_1 on the end conditions corresponding to various delamination lengths are computed without difficulty. These results are shown in Fig. 2. It is seen from Fig. 2 that the effect of the end restraining conditions for this particular case is negligible for delamination lengths varying from 0 to over 90 percent of the total length. Hence, the rotational restraints are not considered in the numerical examples for the general case presented in the next section. Numerical results calculated by using the present computer program for beam-plates having various delamination lengths are also used in the next section to support the program for general case where both the normal and shear stiffness of the core are accounted for.

Buckling loads with effects of normal and shear stiffnesses of the core

For partially checking the derivation and computer program for the general solution for determining the buckling load accounting for the transverse normal and shear resistance from the core, a both ends simply supported face sheet having $l = 31.4$ inches and $D_{11} = 100$ lb-in without the shear resistance from the core is first considered. The results obtained by using the program based on eqn (41) without B_0'' and B_1'' established for the case considered in the last section are compared to the results by using the program written for the general

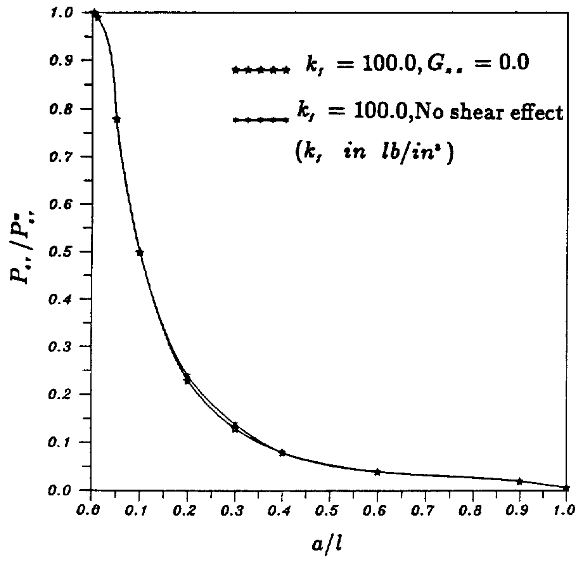

Fig. 3. Checking computational and analysis procedures.

case corresponding to the eqns (36) and (37) for case 3 which are reduced from eqn (40). Close agreements on the buckling load corresponding to various delamination lengths for $k_f = 100$ lb/in^3 are noted in Fig. 3 for results obtained by using these two separately written computer programs. All converged solutions computed using these two computer programs are essentially identical except for $a = 0.2l$ and $0.3l$ where slight discrepancies are shown in Fig. 3. For the line corresponding to $k_f = 100$ and $G_{xz} = 0.0$ in Fig. 3, eqns (36) and (37) are used. We arrive at a converged solution requiring 60 divisions at the debonded region which corresponds to a 120 by 120 buckling determinant for $a = 0.2l$. On the other hand, eqn (41) with $B''_0 = B''_1 = 0$ is used for the line corresponding to $k_f = 100$ and no shear effect for which 60 divisions at the debonded region corresponding to a 60 by 60 buckling determinant for the converged solution for $a = 0.2l$. For $a = 3l$, a 200 by 200 vs a 100 by 100 buckling determinants are used, respectively. Although these discrepancies are negligible, using properly reduced equations for specific cases would not only reduce the computational time but should also give more accurate results. In Fig. 3, P^{w}_{cr} represents the buckling load of the face sheet without debond and shear stiffness of the core. The rapid rate of reduction in the load carrying capacity with the increase of delamination length is consistent to the findings given by Somer (1991).

For the same face sheet used in the last case having a central delamination length of $a = 0.1l$ with $k_f = 100$ lb/in^3, the buckling load corresponding to various values of the transverse shear to normal stiffness ratio of the core is shown in Fig. 4. The symbol s_f for $2G_{xz}/c$ is used to represent the shear stiffness of the core. As expected, the buckling load increases as the shear stiffness increases.

Results on the buckling load for the same face sheet used in the last case corresponding to the core stiffnesses of $k_f = 100$ lb/in^3 with $s_f = 10k_f$ for various delamination length ratios are compared to the $s_f = 0$ case as shown in Fig. 5. From Fig. 5, it may be noted that the effect of shear stiffness is significant for longer delamination length. For this case, the buckling load for $s_f = 10k_f$ has a 38.65 percent increase over the $s_f = 0$ case when the face sheet has a central debond of 70 percent. Based on the results shown in Fig. 4, the effect of s_f to the buckling load would be more pronounced for higher values of the shear stiffness.

CONCLUDING REMARKS

The main objective of the study is to explore the feasibility of using the concept of continuous analysis for determining the buckling load of face sheets delaminated from the core of sandwich beams. The analysis, treating such discontinuous structural components as a continuous system without debonds but subjected to a fictitiously added force system

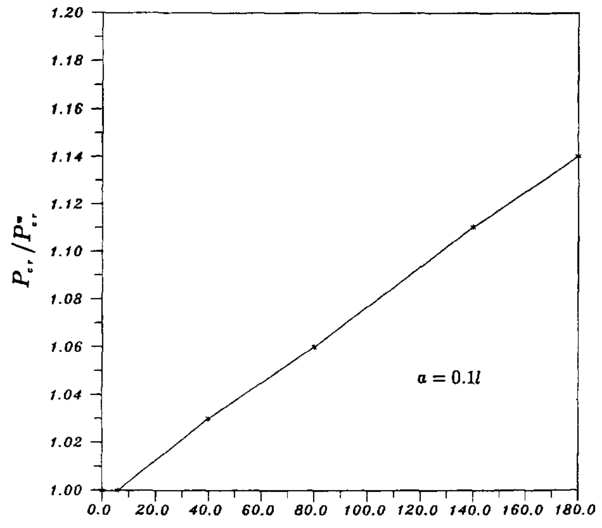

Fig. 4. Effect of shear stiffness of the core for a unit width beam having a central debond.

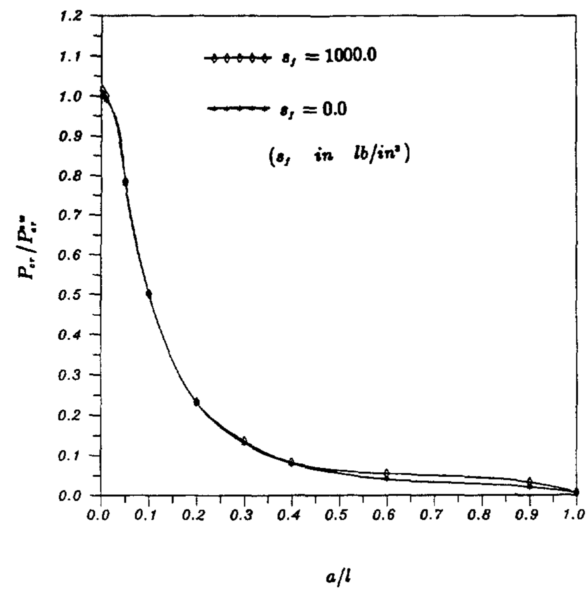

Fig. 5. Critical load of a unit width beam having a central debond.

that causes the net interfacial tractions at the separated regions to vanish, is simple, systematic and effective. The formulation of the problem is general as both the transverse normal as well as the shear stiffnesses of the core are accounted for. Some simple numerical examples are presented for demonstrating the applicability and accuracy of the method of continuous analysis and the use of the Stokes transformation. Numerical results obtained on the basis of the present analysis compare well with existing degenerated cases which provided partial checks and support of the present method of analysis. Fourier series in conjunction with the Stokes transformation used in the analysis provides an effective unified solution for the face sheets having different types of support conditions. In the usual conventional structural analysis, the procedure for solving the problem would divide the entire face sheet into segments with and without debonds. The buckling determinant is then obtained by satisfying continuities in forces and displacements at junctions of adjacent bonded and disbonded regions. The present analysis which deals with the entire region as a whole is more direct than the conventional analysis. Furthermore, in anticipation of extending the analysis to two-dimensional composite plate problems having delaminated regions of arbitrary shapes, the use of the conventional procedure would be very difficult if not impossible while the use of the Fourier series solutions with Stokes transformation in the present continuous analysis procedure does not foreseeably pose difficulties.

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